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The joint universality theorem for a pair of Hurwitz zeta functions

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ABSTRACT

In this paper we investigate the joint functional distribution for a pair of Hurwitz zeta functions $\zeta(s,\alpha_j)$ $(j=1,2)$ in the case that real transcendental numbers α_1 and α_2 satisfy $\alpha_2 \in \mathbb{Q}(\alpha_1)$. Especially we establish the joint universality theorem for these zeta functions.

1. Introduction

As usual, let $s=\sigma+it$ be a complex valuable and $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all natural numbers, non-negative integers, integers, rational numbers, real numbers and complex numbers respectively. In order to state our results, we define some symbols. Let D be the strip $\{s \in \mathbb{C} \mid \frac{1}{2} < \sigma < 1\}$. Let μ be the Lebesgue measure on $\mathbb{R}$ and for $T>0$

$$\nu_T(\cdots) = \frac{1}{T}\mu\bigl\{\tau \in [0,T]\colon \cdots\bigr\},$$

where in place of dots we write some conditions satisfied by τ. For $\sigma>1$ the Rieamnn zeta function

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$\zeta(s)$ is given by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}.$$

In 1975, S.M. Voronin [8] established the remarkable universality theorem for $\zeta(s)$.

Theorem 1. *Let K be a compact subset of D with connected complement and $f(s)$ be a continuous and non-vanishing function on K which is analytic in the interior of K. Then for any $\varepsilon>0$ we have*

$$\liminf_{T\to\infty}\nu_T\left(\max_{s\in K}\left|\zeta(s+i\tau)-f(s)\right|<\varepsilon\right)>0.$$

The set of Dirichlet exponents $\{\log p \mid p\text{: prime}\}$ of $\log\zeta(s)$ is linearly independent over $\mathbb{Q}$. This property plays an essential role in the proof of Theorem 1. In fact, for many zeta functions with Euler products the universality theorems have been established by several mathematicians (see Section 1.6 in the book [7] by J. Steuding). Let $0<\alpha\leqslant 1$ and $\lambda\in\mathbb{R}$ be fixed. For $\sigma>1$ the Lerch zeta function $L(\lambda,\alpha,s)$ is defined by

$$L(\lambda,\alpha,s)=\sum_{m=0}^{\infty}\frac{e(m\lambda)}{(m+\alpha)^s},$$

where $e(x)=e^{2\pi ix}$. Especially when $\lambda\in\mathbb{Z}$, the Lerch zeta function reduces to the Hurwitz zeta function

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s}.$$

We have

$$\zeta(s,1)=\zeta(s),\qquad \zeta\left(s,\frac{1}{2}\right)=\left(2^s-1\right)\zeta(s).$$

Therefore the universal property also holds for these two zeta functions. In 1980s, B. Bagchi [1] and S.M. Gonek [3] independently obtained the universality theorem for ordinary Hurwitz zeta functions.

Theorem 2. *Let K be a compact subset of D with connected complement and $f(s)$ be a continuous function on K which is analytic in the interior of K. Suppose that α is a real transcendental number or a rational number with $\alpha\neq 1,\frac{1}{2}$. Then for any $\varepsilon>0$ we have*

$$\liminf_{T\to\infty}\nu_T\left(\max_{s\in K}\left|\zeta(s+i\tau,\alpha)-f(s)\right|<\varepsilon\right)>0.$$

When α is a transcendental number, the set of Dirichlet exponents $\{\log(m+\alpha)\mid m\geqslant 0\}$ of $\zeta(s,\alpha)$ is linearly independent over $\mathbb{Q}$. Therefore the universal property for $\zeta(s,\alpha)$ follows from the similar argument as in the proof of Theorem 1. When α is a rational number, namely $\alpha=\frac{a}{q}$, we have

$$\zeta\left(s,\frac{a}{q}\right)=q^s\sum_{\chi\ (\mathrm{mod}\ q)}\overline{\chi}(a)L(s,\chi),\tag{1}$$

where $L(s,\chi)$ is the Dirichlet L-function

$$L(s,\chi)=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}\quad(\sigma>1).$$

The universal property for $\zeta(s,\frac{a}{q})$ is obtained from (1) and the following joint universality theorem for Dirichlet L-functions, which was established by Bagchi [2], Gonek [3] and S.M. Voronin [9] independently.

Theorem 3. *For each $1\leqslant j\leqslant r$, let χ_j be a pairwise inequivalent Dirichlet character, K_j be a compact subset of D with connected complement and $f_j(s)$ be a non-vanishing and continuous function on K_j which is analytic in the interior of K_j. Then for any $\varepsilon>0$ we have*

$$\liminf_{T\to\infty}\nu_T\Bigl(\max_{1\leqslant j\leqslant r}\max_{s\in K_j}\bigl|L(s+i\tau,\chi_j)-f_j(s)\bigr|<\varepsilon\Bigr)>0.$$

Recently several mathematicians investigated the joint value distribution of a set of Lerch zeta functions. A. Laurinčikas and K. Matsumoto [4] established the joint universality theorem for the set $\{L(\lambda_j,\alpha_j,s)\mid j=1,\dots,r\}$ in the case that α_j's are real transcendental numbers which are algebraically independent over $\mathbb{Q}$ and λ_j's are rational numbers satisfying some algebraically conditions. Later, T. Nakamura [6] succeeded in proving the joint universality theorem of $\{L(\lambda_j,\alpha_j,s)\}$ for arbitrary real numbers λ_j's. From Theorem 1.3 in [6], we obtain the joint universality theorem for a set of Hurwitz zeta functions.

Theorem 4. *For each $1\leqslant j\leqslant r$, let α_j be a real transcendental number with $0<\alpha_j<1$, K_j be a compact subset of D with connected complement and $f_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j. Suppose that $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then for any $\varepsilon>0$ we have*

$$\liminf_{T\to\infty}\nu_T\Bigl(\max_{1\leqslant j\leqslant r}\max_{s\in K_j}\bigl|\zeta(s+i\tau,\alpha_j)-f_j(s)\bigr|<\varepsilon\Bigr)>0.$$

For α_j's under consideration, it is obvious that a union of the sets of Dirichlet exponents

$$\bigcup_{1\leqslant j\leqslant r}\bigl\{\log(m+\alpha_j)\bigm| m\geqslant 0\bigr\}$$

is linearly independent over $\mathbb{Q}$. The purpose of this paper is to investigate the joint value distribution of a set of Hurwitz zeta functions $\zeta(s,\alpha_j)$ $(1\leqslant j\leqslant r)$ when $\alpha_1,\dots,\alpha_r$ satisfy some algebraic relation over $\mathbb{Q}$. When $r\geqslant 3$, Nakamura [6] has shown that there exist sets of Lerch zeta functions $\{L(\lambda_j,\alpha_j,s)\}$ which do not have the joint universal property. For instance, let α_1 be an arbitrary transcendental real number with $0<\alpha_1<1$. Putting

$$\alpha_2=\frac{\alpha_1}{2}\quad\text{and}\quad\alpha_3=\frac{\alpha_1+1}{2},$$

then we have

$$\zeta(s,\alpha_1)=\frac{1}{2^s}\bigl\{\zeta(s,\alpha_2)+\zeta(s,\alpha_3)\bigr\}.$$

This relation implies that the joint universality theorem does not hold for these three zeta functions.

Now we consider the case that $r = 2$. In the following, assume that

$$0 < \alpha_1, \alpha_2 < 1, \quad \alpha_1 \neq \alpha_2, \quad \alpha_2 \in \mathbb{Q}(\alpha_1). \tag{2}$$

Our main result is the joint universality theorem for $\zeta(s, \alpha_1)$ and $\zeta(s, \alpha_2)$.

Theorem 5. *Suppose that real transcendental numbers α_1 and α_2 satisfy the condition (2). For each $j = 1, 2$, let K_j be a compact subset of D with connected complement and $f_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j. Then for any $\varepsilon > 0$ we have*

$$\liminf_{T\to\infty} \nu_T\Big(\max_{1\leqslant j\leqslant 2} \max_{s\in K_j} \big|\zeta(s+i\tau, \alpha_j) - f_j(s)\big| < \varepsilon\Big) > 0.$$

As corollaries of the theorem, we will establish the joint denseness result and joint functional independence for the pair of Hurwitz zeta functions.

Corollary 1. *Suppose that real transcendental numbers α_1 and α_2 satisfy the condition (2). Let $r \geqslant 1$ and σ_j be a real number with $\frac{1}{2} < \sigma_j < 1$ for each $j = 1, 2$. Then a set*

$$\big\{\big(\zeta(\sigma_1+i\tau, \alpha_1), \zeta(\sigma_2+i\tau, \alpha_2), \ldots, \zeta^{(r-1)}(\sigma_1+i\tau, \alpha_1), \zeta^{(r-1)}(\sigma_2+i\tau, \alpha_2)\big) \in \mathbb{C}^{2r} \bigm| \tau \in \mathbb{R}\big\}$$

is dense in $\mathbb{C}^{2r}$. Namely for any $a_{jk} \in \mathbb{C}$ ($j = 1, 2$, $k = 0, \ldots, r-1$) and $\varepsilon > 0$ we have

$$\liminf_{T\to\infty} \nu_T\Big(\max_{1\leqslant j\leqslant 2} \max_{0\leqslant k\leqslant r-1} \big|\zeta^{(k)}(\sigma_j+i\tau, \alpha_j) - a_{jk}\big| < \varepsilon\Big) > 0.$$

Corollary 2. *Suppose that real transcendental numbers α_1 and α_2 satisfy the condition (2). Let $r \geqslant 1$, $l \geqslant 0$ and F_j ($0 \leqslant j \leqslant l$) be continuous functions on $\mathbb{C}^{2r}$ such that*

$$\sum_{j=0}^{l} s^j F_j\big(\zeta(s, \alpha_1), \zeta(s, \alpha_2), \ldots, \zeta^{(r-1)}(s, \alpha_1), \zeta^{(r-1)}(s, \alpha_2)\big) = 0,$$

holds identically for $s \in \mathbb{C}$. Then $F_j \equiv 0$ for all $0 \leqslant j \leqslant l$.

In Section 2 we will study the linearly independence of the sets of Dirichlet exponents. In Section 3 we will prove Theorem 5. In Section 4 we will deduce the corollaries.

2. Linearly independence of Dirichlet exponents

First we consider the case that α_2 is a linear form of α_1 over $\mathbb{Q}$. Namely,

$$\alpha_2 = \frac{p}{q}\alpha_1 + \frac{a}{b}, \tag{3}$$

where q, b are positive integers and p, a are integers which satisfy $(p, q) = 1$ and $(a, b) = 1$.

Lemma 1. *Suppose that real transcendental numbers* α_1 *and* α_2 *satisfy the conditions* (2) *and* (3).

(1) *When* $q \not\equiv 0 \pmod b$, *then the set*

$$\bigcup_{j=1,2} \{\log(m+\alpha_j) \mid m \geqslant 0\}$$

is linearly independent over $\mathbb{Q}$.

(2) *When* $q \equiv 0 \pmod b$ *and* $p < 0$, *there exists* $m_0 > 0$ *such that the set*

$$\{\log(m+\alpha_1) \mid m \geqslant 0\} \cup \{\log(m+\alpha_2) \mid m \geqslant m_0\}$$

is linearly independent over $\mathbb{Q}$.

(3) *When* $q \equiv 0 \pmod b$ *and* $p > 0$, *there exist integers* $0 \leqslant c_1 < p$, $0 \leqslant d_1 < q$ *and* n_0 *which have the following properties*:

(a) *The set*

$$\{\log(m+\alpha_1) \mid m \geqslant 0,\ m \not\equiv d_1 \pmod q\} \cup \{\log(m+\alpha_2) \mid m \geqslant 0\}$$

is linearly independent over $\mathbb{Q}$.

(b) *If we set* $\Lambda_1 = \{\log(m+\alpha_1) \mid m \geqslant 0,\ m \equiv d_1 \pmod q\} = \{\log(qn+d_1+\alpha_1) \mid n \geqslant 0\}$ *and* $\Lambda_2 = \{\log(pn+c_1+\alpha_1) \mid n \geqslant n_0\}$, *then a bijection from* Λ_2 *to* Λ_1 *is given by*

$$\log(pn+c_1+\alpha_2) \mapsto \log\bigl(q(n-n_0)+d_1+\alpha_1\bigr).$$

Namely, for any $x > 0$ *and* $s \in \mathbb{C}$, *we have*

$$\sum_{n_0 \leqslant n \leqslant x} \frac{1}{(pn+c_1+\alpha_2)^s} = \left(\frac{q}{p}\right)^s \sum_{0 \leqslant n \leqslant x-n_0} \frac{1}{(qn+d_1+\alpha_1)^s}. \tag{4}$$

Proof. Consider an equation

$$\frac{\prod_i (h_i+\alpha_1)^{a_i}}{\prod_j (k_j+\alpha_1)^{b_j}} = \frac{\prod_I (H_I+\alpha_2)^{A_I}}{\prod_J (K_J+\alpha_2)^{B_J}}, \tag{5}$$

where h_i and k_j are non-negative integers which are different each other, H_I and K_J are also non-negative integers which are different each other, and a_i, b_j, A_I, and B_J are positive integers. To prove the lemma, we investigate conditions that (5) becomes an identity of α_1. By (3), Eq. (5) is rewritten as follows

$$\frac{\prod_i (h_i+\alpha_1)^{a_i}}{\prod_j (k_j+\alpha_1)^{b_j}} = \left(\frac{p}{q}\right)^{\sum A_I - \sum B_J} \frac{\prod_I \{\alpha_1 + \frac{q}{p}(H_I + \frac{a}{b})\}^{A_I}}{\prod_J \{\alpha_1 + \frac{q}{p}(K_J + \frac{a}{b})\}^{B_J}}.$$

Comparing the both sides, it reduces that $\sum A_I = \sum B_J$ and

$$\frac{q}{p}\left(k + \frac{a}{b}\right) \in \mathbb{N}_0 \quad \text{for } k = H_I, K_J. \tag{6}$$

Since $(p,q) = 1$, (6) is equivalent to

$$q \equiv 0 \pmod b, \quad \text{and} \quad a + bk \equiv 0 \pmod p, \tag{7}$$

for all $k = H_I$ and K_J. If $q \not\equiv 0 \pmod b$, then (5) holds only in the case that all a_i, b_j, A_I and B_J are 0. Therefore the first statement of the lemma is obtained. Next we consider the case that $q \equiv 0 \pmod b$. If $p < 0$, then there are at most finitely many integers k which satisfy (6). Therefore the second statement of the lemma is obtained. Now we consider the case that $p > 0$. Since $(b, p) = 1$, the congruence in (7) holds for an integer k satisfying $k \equiv -a\bar{b} \pmod p$, where $\bar{b}$ is the inverse of b in the residue class group $(\mathbb{Z}/p\mathbb{Z})^\times$. Let c_1 be an integer satisfying $0 \leqslant c_1 < p$ and $c_1 \equiv -a\bar{b} \pmod p$. Then all $k \geqslant 0$ satisfying (7) are given by $pn + c_1$ for $n \geqslant 0$. For such $k = pn + c_1$ we have

$$\begin{aligned}
\log(k + \alpha_2) &= \log\left(pn + c_1 + \frac{p}{q}\alpha_1 + \frac{a}{b}\right) \\
&= \log\left(qn + \frac{q}{b}\frac{bc_1 + a}{p} + \alpha_1\right) + \log\left(\frac{p}{q}\right) \\
&= \log\left(q(n - n_0) + \left(qn_0 + \frac{q}{b}\frac{bc_1 + a}{p}\right) + \alpha_1\right) + \log\left(\frac{p}{q}\right).
\end{aligned} \tag{8}$$

Since $c_1 \equiv -a\bar{b} \pmod p$, the quotient $q(bc_1 + a)b^{-1}p^{-1}$ is an integer. If we choose a suitable n_0 then $d_1 = qn_0 + q(bc_1 + a)b^{-1}p^{-1}$ satisfies $0 \leqslant d_1 < q$. Eq. (8) gives the bijection from Λ_2 to Λ_1 and the relation (4). This completes the proof of the lemma. □

Next we consider the remaining case. Let $P(x) = \sum_{0 \leqslant m \leqslant M} a_m x^m$ and $Q(x) = \sum_{0 \leqslant n \leqslant N} b_n x^n$ be polynomials with integral coefficients. Suppose that $P(x)$ and $Q(x)$ are prime each other and

$$\alpha_2 = \frac{P(\alpha_1)}{Q(\alpha_1)}. \tag{9}$$

Lemma 2. *Let α_1 and α_2 be transcendental real numbers satisfying* (2) *and* (9)*. Suppose that at least one of inequalities $M \geqslant 2$ or $N \geqslant 1$ holds. Then there exists $m_0 > 0$ such that the set*

$$\{\log(m + \alpha_1) \mid m \geqslant 0\} \cup \{\log(m + \alpha_2) \mid m \geqslant m_0\}$$

is linearly independent over $\mathbb{Q}$.

Proof. As in the proof of Lemma 1, we consider Eq. (5). For $k \in \mathbb{N}_0$, we set $f_k(x) = P(x) + kQ(x)$. Then (5) is rewritten as follows

$$\frac{\prod_i (h_i + \alpha_1)^{a_i}}{\prod_j (k_j + \alpha_1)^{b_j}} = Q(\alpha_1)^{\sum B_J - \sum A_I} \frac{\prod_I f_{H_I}(\alpha_1)^{A_I}}{\prod_J f_{K_J}(\alpha_1)^{B_J}}. \tag{10}$$

Since $P(x)$ and $Q(x)$ are prime each other, $f_k(x)$ and $f_l(x)$ have no common divisor when $k \neq l$. Therefore Eq. (10) becomes an identity of α_1 only in the case that all solutions of an equation $f_k(x) = 0$ are non-positive integers for each $k = H_I, K_J$. Lemma 2 follows from the lemma below. □

Lemma 3. *Suppose that at least one of inequalities $N \geqslant 1$ or $M \geqslant 2$ holds. For $k \in \mathbb{N}_0$ put $f_k(x) = P(x) + kQ(x)$. There are at most finitely many numbers of k for which all solutions of $f_k(x) = 0$ are non-positive integers only.*

Proof. Suppose that $N \geqslant M$. Put $a_m = 0$ for $M < m \leqslant N$, then $f_k(x) = (a_N + kb_N)x^N + \cdots + (a_0 + kb_0)$. If $f_k(x) = 0$ has non-positive integer solutions $-l_1, \ldots, -l_N$ only, then

$$l_1 + \cdots + l_N = \frac{a_{N-1} + kb_{N-1}}{a_N + kb_N}. \tag{11}$$

Since $b_N \neq 0$, the right-hand side of (11) is bounded as a function of k. Remark that $f_k(x) = 0$ and $f_{k'}(x) = 0$ have no common integral solutions if $k \neq k'$. Therefore there are at most finitely many numbers of vectors $(k, l_1, \ldots, l_N)$ satisfying (11). Next we consider the case that $M \geqslant N + 2$, namely $f_k(x) = a_M x^M + a_{M-1}x^{M-1} + \cdots + (a_0 + kb_0)$. Assume that $f_k(x) = 0$ has non-positive integer solutions $-l_1, \ldots, -l_M$, then

$$l_1 + \cdots + l_M = \frac{a_{M-1}}{a_M}. \tag{12}$$

It is obvious that there are at most finitely many numbers of vectors $(k, l_1, \ldots, l_M)$ satisfying (12). Lastly we consider the case that $M = N + 1$ and $N \geqslant 1$. Assume that $f_k(x) = a_M x^M + (a_{M-1} + kb_{M-1})x^{M-1} + (a_0 + kb_0) = 0$ has non-positive integer solutions $-l_1, \ldots, -l_M$. Then we have

$$\prod_{i=1}^{M} l_i = \frac{a_0 + kb_0}{a_M} \tag{13}$$

and

$$\sum_{i=1}^{M} \biggl(\prod_{j \neq i} l_j \biggr) = \frac{a_1 + kb_1}{a_M}. \tag{14}$$

If $b_0 a_M^{-1} \leqslant 0$, then the number of vectors $(k, l_1, \ldots, l_M)$ satisfying (13) is at most finite. Therefore we may assume that a_M and b_0 have the same sign. By the similar argument, (14) allows us to assume that a_M and b_1 have the same sign. From (13) and (14) we have

$$\sum_{i=1}^{N} \frac{1}{l_i} = \frac{a_1 + kb_1}{a_0 + kb_0}. \tag{15}$$

Since b_0 and b_1 have the same sign, the right-hand side of (15) is bounded as a function of k. Therefore the number of vectors $(k, l_1, \ldots, l_M)$ satisfying (15) is at most finite. This completes the proof of Lemma 3. □

3. Proof of Theorem 5

In this section we prove Theorem 5. First we quote some fundamental lemmas. The following lemma is obtained in [5] as Lemma 2.5.

Lemma 4. *Let K and U be simply connected compact subsets of $\mathbb{C}$ such that K is included in the interior of U. Let $f(s)$ be an analytic function on U. If we set*

$$a = \iint_U \bigl| f(s) \bigr|^s \, d\sigma \, dt,$$

then

$$\max_{s\in K}\left|f(s)\right| \leqslant c\sqrt{a},$$

where $c = c(U, K)$ is a positive constant depends only on U and K.

Next we recall the Mergelyan's theorem, which is a complex analogue of the Weierstrass's approximation theorem.

Lemma 5. *Let K be a compact subset of $\mathbb{C}$ with connected complement and $f(s)$ be a continuous function on K which is analytic in the interior of K. Then for any $\varepsilon > 0$ there exists a polynomial $p(s)$ such that*

$$\max_{s\in K}\left|f(s) - p(s)\right| < \varepsilon.$$

The following lemma is the well-known Kronecker's approximation theorem, which is the key in the proof of universality theorems.

Lemma 6. *(See [10, Theorem A8.1 and Theorem A8.3].) Let $\beta_1, \dots, \beta_N$ be real numbers which are linearly independent over $\mathbb{Q}$. Let γ be a closed subregion of N-dimensional unit cube with Jordan volume Γ. For $T > 0$ we set*

$$I_\gamma(T) = \left\{\tau \in [0, T] \;\middle|\; (\beta_1\tau, \dots, \beta_N\tau) \in \gamma \pmod 1\right\}.$$

(1) *The set $I_\gamma(T)$ has a positive density*

$$\lim_{T\to\infty} \frac{\mu(I_\gamma(T))}{T} = \Gamma.$$

(2) *Let Ω be a family of complex valued continuous functions on γ. If Ω is uniformly bounded and equicontinuous, then the following relation holds uniformly for $f \in \Omega$:*

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_{I_\gamma(T)} f(\beta_1\tau, \dots, \beta_N\tau)\, d\tau = \int\limits_{\gamma} f\, dx_1 \cdots dx_N.$$

Let $\Lambda = \{\lambda\}$ be a monotone increasing sequence of real numbers tending to ∞. For $x > 0$ we set

$$N_\Lambda(x) = \sum_{\substack{\lambda\in\Lambda\\ \lambda\leqslant x}} 1.$$

In [3], Gonek established the fundamental denseness lemma.

Lemma 7. *Suppose that*

$$N_\Lambda(x) \ll e^x$$

and that for any fixed $c > 0$

$$\left|N_\Lambda\left(x+\frac{c}{x^2}\right)-N_\Lambda(x)\right| \gg \frac{e^x}{x^3}.$$

Let K be a simply connected compact subset in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ and $f(s)$ be a continuous function on K which is analytic in the interior of K. Then for any $\nu > 0$ there exists $\rho_0 > 0$ such that if $\rho \geqslant \rho_0$ there are numbers $\theta_\lambda \in \mathbb{R}$ for which

$$\max_{s\in K}\left|f(s)-\sum_{\substack{\nu<e^\lambda\leqslant\rho\\ \lambda\in\Lambda}} e(\theta_\lambda)e^{-\lambda s}\right| \ll_K \nu^{-\frac{1}{2}}.$$

Lastly we quote the approximate formula for the Hurwitz zeta function.

Lemma 8. *Let $T > 0$ and $0 < \sigma_0 < 2$. For $\sigma_0 \leqslant \sigma \leqslant 2$ and $2\pi \leqslant |t| \leqslant \pi T$ we have*

$$\zeta(s,\alpha)=\sum_{m\leqslant T}\frac{1}{(m+\alpha)^s}+\frac{T^{1-s}}{s-1}+O\bigl(T^{-\sigma}\bigr).$$

Let K_1, K_2, $f_1(s)$, $f_2(s)$ and ε be taken as in Theorem 1. Let K_3 be a simply connected compact subset of D such that the union $K_1 \cup K_2$ is included in the interior of K_3. By Lemma 5 there are polynomials $p_1(s)$ and $p_2(s)$ satisfying

$$\max_{j=1,2}\max_{s\in K_j}\bigl|f_j(s)-p_j(s)\bigr| < \varepsilon. \tag{16}$$

Remark that these polynomials $p_j(s)$'s satisfy the condition in Lemma 7 on the set K_3. Let $z > 0$. We calculate the second moment

$$I=\int_T^{2T}\iint_{K_3}\sum_{j=1}^{2}\left|\zeta(s+i\tau,\alpha_j)-\sum_{m_j\leqslant z}\frac{1}{(m_j+\alpha_j)^{s+i\tau}}\right|^2 d\sigma\,dt\,d\tau.$$

Taking into account Lemma 8, we have

$$\begin{aligned}I &\ll T\int_T^{2T}\sum_{j=1,2}\left|\sum_{z<m_j\leqslant T}\frac{1}{(m_j+\alpha_j)^{\sigma_3+i\tau}}\right|^2 d\tau + T^{1-2\sigma_3+\varepsilon'}\\ &\ll T\bigl(T^{1-2\sigma_3+\varepsilon'}+z^{1-2\sigma_3+\varepsilon'}\bigr),\end{aligned} \tag{17}$$

where $\sigma_3 = \min\{\Re s \mid s \in K_3\} > \frac{1}{2}$, ε' is an arbitrary positive number and the implicit constant depends only on ε' and K_3. Combining (17) and Lemma 4, we obtain the lemma below.

Lemma 9. *Let ε_1 and ε_2 be small positive numbers. Consider a set*

$$A_T=\left\{\tau\in[T,2T]\;\middle|\;\max_{j=1,2}\max_{s\in K_j}\left|\zeta(s+i\tau,\alpha_j)-\sum_{m\leqslant z}\frac{1}{(m+\alpha_j)^{s+i\tau}}\right|<\varepsilon_1\right\}.$$

There exists $z_0 > 0$ such that if $z \geqslant z_0$ then

$$\liminf_{T \to \infty} \frac{\mu(A_T)}{T} > 1 - \varepsilon_2.$$

Now we suppose that α_1 and α_2 satisfy the hypothesis in Lemma 1(3). Namely

$$\alpha_2 = \frac{p}{q}\alpha_1 + \frac{a}{b}, \quad q \equiv 0 \ (\mathrm{mod}\ b), \ \text{and}\ p > 0. \tag{18}$$

Let c_1, d_1 and n_0 be the integers as in Lemma 1(3). Let Λ_1 and Λ_2 be the sets as in Lemma 1(3). Further we put

$$\Lambda_3 = \bigl\{\log(m + \alpha_1) \bigm| m \geqslant 0, \ m \not\equiv d_1 \ (\mathrm{mod}\ q)\bigr\},$$

and

$$\Lambda_4 = \bigl\{\log(m + \alpha_2) \bigm| m \geqslant 0, \ m \not\equiv c_1 \ (\mathrm{mod}\ p)\bigr\}.$$

These sets Λ_j $(2 \leqslant j \leqslant 4)$ satisfy the hypothesis of Lemma 7. Let ν be a sufficiently large number such that

$$\nu^{-\frac{1}{2}} < \frac{\varepsilon}{4c(K_3)pq} \tag{19}$$

holds, where $c(K_3)$ is the implicit constant in Lemma 7. By Lemma 7, there exists $\rho_0 > 0$ such that if $\rho \geqslant \rho_0$ there are sequences θ_m, η_m, $\theta_n \in \mathbb{R}$ for which

$$\max_{s \in K_3} \left| \sum_{n_0 \leqslant n \leqslant \frac{\nu}{p}} \frac{1}{(pn + c_1 + \alpha_2)^s} + \sum_{\frac{\nu}{p} < n \leqslant \frac{\rho}{p}} \frac{e(\theta_n)}{(pn + c_1 + \alpha_2)^s} \right| < \frac{\varepsilon}{2}, \tag{20}$$

$$\max_{s \in K_3} \left| p_1(s) - \sum_{\substack{m \leqslant \frac{q}{p}\nu \\ m \not\equiv d_1\ (q)}} \frac{1}{(m + \alpha_1)^s} - \sum_{\substack{\frac{q}{p}\nu < m \leqslant \frac{q}{p}\rho \\ m \not\equiv d_1\ (q)}} \frac{e(\theta_m)}{(m + \alpha_1)^s} \right| < \frac{\varepsilon}{2}, \tag{21}$$

and

$$\max_{s \in K_3} \left| p_2(s) - \sum_{n < n_0} \frac{1}{(pn + c_1 + \alpha_2)^s} - \sum_{\substack{m \leqslant \nu \\ m \not\equiv c_1\ (p)}} \frac{1}{(m + \alpha_2)^s} - \sum_{\substack{\nu < m \leqslant \rho \\ m \not\equiv c_1\ (p)}} \frac{e(\eta_m)}{(m + \alpha_2)^s} \right| < \frac{\varepsilon}{2} \tag{22}$$

hold. With respect to the set Λ_1, Eq. (4) in Lemma 1, (19) and (20) imply

$$\max_{s \in K_3} \left| \sum_{n \leqslant \frac{\nu}{p} - n_0} \frac{1}{(qn + d_1 + \alpha_1)^s} + \sum_{\frac{\nu}{p} - n_0 < n \leqslant \frac{\rho}{p} - n_0} \frac{e(\theta_n)}{(qn + d_1 + \alpha_1)^s} \right| < \frac{\varepsilon}{2}. \tag{23}$$

Let $\theta_m = 0$ for $0 \leqslant m \leqslant \frac{q}{p}\nu$ with $m \not\equiv d_1 \pmod q$, $\eta_m = 0$ for $0 \leqslant m \leqslant \nu$ with $m \not\equiv c_1 \pmod p$, and $\theta_n = 0$ for $0 \leqslant n \leqslant \frac{\nu}{p}$. For $\delta > 0$ define a set B_T consisting of $\tau \in [T, 2T]$ for which inequalities

$$\left\| -\tau \frac{\log(m+\alpha_1)}{2\pi} - \theta_m \right\| \leqslant \delta \quad \left(0 \leqslant m \leqslant \frac{q}{p}\rho,\ m \not\equiv d_1 \ (\mathrm{mod}\ q)\right),$$

$$\left\| -\tau \frac{\log(m+\alpha_2)}{2\pi} - \eta_m \right\| \leqslant \delta \quad \left(0 \leqslant m \leqslant \rho,\ m \not\equiv c_1 \ (\mathrm{mod}\ p)\right),$$

and

$$\left\| -\tau \frac{\log(pn+c_1+\alpha_2)}{2\pi} - \theta_n \right\| \leqslant \delta \quad \left(0 \leqslant n \leqslant \frac{1}{p}\rho\right),$$

hold, where $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$. By continuity of the sums in (20)–(23), if we choose δ sufficiently small, then for any $\tau \in B_T$ inequalities

$$\max_{s\in K_3} \left| \sum_{n_0 \leqslant n \leqslant \frac{\rho}{p}} \frac{1}{(pn+c_1+\alpha_2)^{s+i\tau}} \right| < \varepsilon, \tag{24}$$

$$\max_{s\in K_3} \left| p_1(s) - \sum_{\substack{m \leqslant \frac{q}{p}\rho \\ m \not\equiv d_1\ (q)}} \frac{1}{(m+\alpha_1)^{s+i\tau}} \right| < \varepsilon, \tag{25}$$

$$\max_{s\in K_3} \left| p_2(s) - \sum_{n<n_0} \frac{1}{(pn+c_1+\alpha_2)^{s+i\tau}} - \sum_{\substack{m \leqslant \rho \\ m \not\equiv c_1\ (p)}} \frac{1}{(m+\alpha_2)^{s+i\tau}} \right| < \varepsilon, \tag{26}$$

and

$$\max_{s\in K_3} \left| \sum_{n \leqslant \frac{\rho}{p} - n_0} \frac{1}{(qn+d_1+\alpha_1)^{s+i\tau}} \right| < \varepsilon \tag{27}$$

hold. Now we apply Lemma 6(1). Since the set $\Lambda_2 \cup \Lambda_3 \cup \Lambda_4$ is linearly independent over $\mathbb{Q}$, the set B_T has a positive density

$$\lim_{T\to\infty} \frac{\mu(B_T)}{T} = (2\delta)^{(1+\frac{p-1}{p})\rho}. \tag{28}$$

Assume that $z > \rho$. Calculate the second mean moment with respect to the set Λ_3

$$J = \int_{B_T} \iint_{K_3} \left| \sum_{\substack{\frac{1}{p}\rho < m \leqslant z \\ m \not\equiv d_1\ (q)}} \frac{1}{(m+\alpha_1)^{s+i\tau}} \right|^2 d\sigma\, dt\, d\tau.$$

Applying Lemma 6(2), we have

$$J \ll \mu(B_T)\rho^{1-2\sigma_3+\varepsilon'},$$

where ε' be an arbitrary small positive number and the implicit constant depends on K_3. For the sets Λ_2 and Λ_4 we have the similar estimates. Taking into account Lemma 4, (4), (28) and these inequalities, we obtain the next lemma.

Lemma 10. *Let $\rho \geqslant \rho_0$. Define a subset B'_T consisting of $\tau \in B_T$ such that inequalities*

$$\max_{s\in K_3}\Bigg|\sum_{\substack{\frac{q}{p}\rho<m\leqslant z\\ m\not\equiv d_1\,(q)}}\frac{1}{(m+\alpha_1)^{s+i\tau}}\Bigg|<\varepsilon,$$

$$\max_{s\in K_3}\Bigg|\sum_{\substack{\rho<m\leqslant z\\ m\not\equiv c_1\,(p)}}\frac{1}{(m+\alpha_2)^{s+i\tau}}\Bigg|<\varepsilon,$$

$$\max_{s\in K_3}\Bigg|\sum_{\frac{1}{p}\rho<n\leqslant\frac{z-c_1}{p}}\frac{1}{(pn+c_1+\alpha_2)^{s+i\tau}}\Bigg|<\varepsilon,$$

and

$$\max_{s\in K_3}\Bigg|\sum_{\frac{1}{p}\rho<n\leqslant\frac{z-c_1}{p}}\frac{1}{(qn+d_1+\alpha_1)^{s+i\tau}}\Bigg|<\varepsilon$$

hold uniformly for $z>\rho$. There exists $\rho_1>\rho_0$ such that if $\rho\geqslant\rho_1$ then

$$\liminf_{T\to\infty}\frac{\mu(B'_T)}{T}>\frac{1}{2}(2\delta)^{(1+\frac{p-1}{p})\rho}.$$

Now we take

$$\varepsilon_1=\varepsilon,\quad\text{and}\quad\varepsilon_2=\frac{1}{4}(2\delta)^{(1+\frac{p-1}{p})\rho}$$

in Lemma 9. Let $\rho\geqslant\rho_1$ and $z\geqslant\max\{\rho,z_0\}$ be fixed. By Lemma 9 and Lemma 10

$$\liminf_{T\to\infty}\frac{\mu(A_T\cap B'_T)}{T}>\frac{1}{4}(2\delta)^{(1+\frac{p-1}{p})\rho}.$$

For $\tau\in A_T\cap B'_T$, Lemma 9, Lemma 10, (16) and (24)–(27) imply

$$\max_{j=1,2}\max_{s\in K_j}\big|\zeta(s+i\tau,\alpha_j)-f_j(s)\big|<6\varepsilon.$$

This completes the proof of the theorem in the case (18).

Next we prove the theorem in the remaining case. By Lemma 1(1), (2) and Lemma 2, we may assume that there exists $m_0>0$ such that the set

$$\big\{\log(m+\alpha_1)\bigm| m\geqslant 0\big\}\cup\big\{\log(m+\alpha_2)\bigm| m\geqslant m_0\big\}$$

is linearly independent over $\mathbb{Q}$. Put $\Lambda_5=\{\log(m_1+\alpha_1)\mid m_1\geqslant 0\}$ and $\Lambda_6=\{\log(m_2+\alpha_2)\mid m_2\geqslant m_0\}$, then these two sets satisfy the hypothesis in Lemma 7. Let $\nu>0$ be a sufficiently large number satisfying

$$\nu^{-\frac{1}{2}}<\frac{\varepsilon}{2C(K_3)}.$$

By Lemma 7, there exists $\rho_0 > 0$ such that if $\rho \geqslant \rho_0$ there are sequences θ_m and η_m $(\nu < m_j \leqslant \rho)$ such that

$$\max_{s\in K_3}\left|p_1(s) - \sum_{0\leqslant m\leqslant \nu}\frac{1}{(m+\alpha_1)^s} - \sum_{\nu<m\leqslant\rho}\frac{e(\theta_m)}{(m+\alpha_1)^s}\right| < \frac{\varepsilon}{2} \tag{29}$$

and

$$\max_{s\in K_3}\left|p_2(s) - \sum_{0\leqslant m\leqslant \nu}\frac{1}{(m+\alpha_2)^s} - \sum_{\nu<m\leqslant\rho}\frac{e(\eta_m)}{(m+\alpha_2)^s}\right| < \frac{\varepsilon}{2} \tag{30}$$

hold. Now we put $\theta_m = 0$ and $\eta_m = 0$ for $0 \leqslant m \leqslant \nu$. Let $n_1, \dots, n_r$ be integers such that $0 \leqslant n_j < m_0$ and the space consisting of the set

$$\bigl\{\log(m+\alpha_1)\bigm| m \geqslant 0\bigr\} \cup \bigl\{\log(n_j+\alpha_2)\bigm| 1 \leqslant j \leqslant r\bigr\}$$

over $\mathbb{Q}$ contains all $\log(m+\alpha_2)$ $(0 \leqslant m < m_0)$. For $\delta > 0$ we define a set C_T of $\tau \in [T, 2T]$ such that inequalities

$$\left\|-\tau\frac{\log(m+\alpha_1)}{2\pi} - \theta_m\right\| \leqslant \delta \quad (0 \leqslant m \leqslant \rho),$$

$$\left\|-\tau\frac{\log(m+\alpha_2)}{2\pi} - \eta_m\right\| \leqslant \delta \quad (m_0 \leqslant m \leqslant \rho),$$

and

$$\left\|-\tau\frac{\log(n_j+\alpha_2)}{2\pi}\right\| \leqslant \delta \quad (1 \leqslant j \leqslant r) \tag{31}$$

hold. By the definition of n_j's, (31) implies that

$$\left\|-\tau\frac{\log(m+\alpha_2)}{2\pi}\right\| \leqslant c(m_0)\delta \quad (0 \leqslant m < m_0),$$

where $c(m_0)$ is a positive constant depends on m_0. By the continuity of the sums in (29) and (30), if δ is sufficiently small then for any $\tau \in C_T$ we have

$$\max_{s\in K_3}\left|p_1(s) - \sum_{0\leqslant m\leqslant\rho}\frac{1}{(m+\alpha_1)^{s+i\tau}}\right| < \varepsilon \tag{32}$$

and

$$\max_{s\in K_3}\left|p_2(s) - \sum_{0\leqslant m\leqslant\rho}\frac{1}{(m+\alpha_2)^{s+i\tau}}\right| < \varepsilon. \tag{33}$$

Now we apply Lemma 6(1). Since the set $\Lambda_5 \cup \Lambda_6 \cup \{\log(n_j + \alpha_2) \mid 1 \leqslant j \leqslant r\}$ is linearly independent over $\mathbb{Q}$, the set C_T has a positive density

$$\lim_{T\to\infty} \frac{\mu(C_T)}{T} = (2\delta)^{2\rho-m_0+r}. \tag{34}$$

Assume that $z > \rho$. Calculate the second mean moment with respect to the set Λ_5

$$L = \int\limits_{C_T} \iint\limits_{K_3} \left| \sum_{\rho<m\leqslant z} \frac{1}{(m+\alpha_1)^{s+i\tau}} \right|^2 d\sigma\, dt\, d\tau.$$

Applying Lemma 6(2), we have

$$L \ll \mu(B_T)\rho^{1-2\sigma_3+\varepsilon'},$$

where ε' be an arbitrary small positive number and the implicit constant depends on K_3. For the set Λ_6 we also have the similar estimates. Taking into account Lemma 4, (34) and these inequalities, we obtain the next lemma.

Lemma 11. *Let $\rho \geqslant \rho_0$. Define a subset C'_T consisting of $\tau \in C_T$ such that inequality*

$$\max_{j=1,2} \max_{s\in K_3} \left| \sum_{\rho<m\leqslant z} \frac{1}{(m+\alpha_j)^{s+i\tau}} \right| < \varepsilon,$$

holds uniformly for $z > \rho$. There exists $\rho_2 > \rho_0$ such that if $\rho \geqslant \rho_2$ then

$$\liminf_{T\to\infty} \frac{\mu(C'_T)}{T} > \frac{1}{2}(2\delta)^{2\rho-m_0+r}.$$

Now we take

$$\varepsilon_1 = \varepsilon, \quad \text{and} \quad \varepsilon_2 = \frac{1}{4}(2\delta)^{2\rho-m_0+r}$$

in Lemma 9. Let $\rho \geqslant \rho_2$ and $z \geqslant \max\{\rho, z_0\}$ be fixed. By Lemma 9 and Lemma 11

$$\liminf_{T\to\infty} \frac{\mu(A_T \cap C'_T)}{T} > \frac{1}{4}(2\delta)^{2\rho-m_0+r}.$$

For $\tau \in A_T \cap C'_T$, Lemma 9, Lemma 11, (16), (32) and (33) imply

$$\max_{j=1,2} \max_{s\in K_j} \left| \zeta(s+i\tau, \alpha_j) - f_j(s) \right| < 5\varepsilon.$$

This completes the proof of the theorem.

4. Proof of corollaries

First we prove Corollary 1. For each $j=1,2$ we put

$$q_j(s)=\sum_{k=0}^{r-1}\frac{a_{jk}}{k!}(s-\sigma_j)^k.$$

Then we have $q_j(s)^{(k)}|_{s=\sigma_j}=a_{jk}$ for $j=1,2$ and $k=0,\dots,r-1$. Let δ be a positive number such that sets $K_j=\{s\in\mathbb{C}\mid |s-\sigma_j|\leqslant\delta\}$ are contained in D. By Theorem 1, if we define a set

$$D_T=\left\{\tau\in[0,T]\;\Big|\;\max_{j=1,2}\max_{s\in K_j}\bigl|\zeta(s+i\tau,\alpha_j)-q_j(s)\bigr|<\frac{2\pi\delta^r\varepsilon}{r!}\right\}$$

then D_T has a positive lower density. Using the Cauchy integral formula and definition of $q_j(s)$'s, for any $\tau\in D_T$ we have $|\zeta^{(k)}(\sigma_j+i\tau,\alpha_j)-a_{jk}|<\varepsilon$ for each $j=1,2$ and $k=0,\dots,r-1$. This completes the proof of the corollary.

Next we prove Corollary 2. Let F_j $(0\leqslant j\leqslant l)$ be continuous functions as in Corollary 2. Assume that $F_l\not\equiv 0$. Then there exists $\delta>0$ and the open subset $U\subset\mathbb{C}^{2r}$ such that

$$\bigl|F_l(z)\bigr|>\delta\quad\text{for all } z\in U.$$

Let $1/2<\sigma_0<1$ be fixed. By Corollary 1, there exists a sequence t_n tending to ∞ such that

$$\bigl(\zeta(s_n,\alpha_1),\zeta(s_n,\alpha_2),\dots,\zeta^{(r-1)}(s_n,\alpha_1),\zeta^{(r-1)}(s_n,\alpha_2)\bigr)\in U,$$

where we set $s_n=\sigma_0+it_n$. Then we have

$$\left|\sum_{j=0}^{l}s_n^jF_j\bigl(\zeta(s_n,\alpha_1),\zeta(s_n,\alpha_2),\dots,\zeta^{(r-1)}(s_n,\alpha_1),\zeta^{(r-1)}(s_n,\alpha_2)\bigr)\right|\gg t_n^l\delta.$$

This contradicts the hypothesis of F_j's. Therefore we obtain the corollary.

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References

[1] B. Bagchi, The statistical behavior and universality properties of the Riemann zeta-function and other allied Dirichlet series, PhD thesis, Indian Statistical Institute, Calcutta, 1981.
[2] B. Bagchi, A joint universality theorem for Dirichlet L-functions, Math. Z. 181 (3) (1982) 319–334.
[3] S.M. Gonek, Analytic properties of zeta and L-functions, PhD thesis, Univ. of Michigan, 1979.
[4] A. Laurinčikas, K. Matsumoto, Joint value-distribution theorems on Lerch zeta-functions. II, Math. J. 46 (3) (2006) 271–286 (in Lithuanian).
[5] H. Mishou, H. Nagoshi, Functional distribution of $L(s,\chi_d)$ with real characters and denseness of quadratic class numbers, Trans. Amer. Math. Soc. 358 (10) (2006) 4343–4366.
[6] T. Nakamura, The existence and the non-existence of joint t-universality for Lerch zeta functions, J. Number Theory 125 (2) (2007) 424–441.
[7] J. Steuding, Value-Distribution of L-Functions and Allied Zeta-Functions – with an Emphasis on Aspects of Universality, Habilitationsschrift, J.W. Goethe-Universität, Frankfurt, 2003.

[8] S.M. Voronin, Theorem on the universality of the Riemann zeta function, Izv. Acad. Nauk. SSSR Ser. Mat. 39 (1975) 475–486 (in Russian); Math. USSR Izv. 9 (1975) 443–453.
[9] S.M. Voronin, Analytic properties of Dirichlet generating functions of arithmetic objects, Math. Notes 24 (6) (1978) 966–969.
[10] A.A. Karatsuba, S.M. Voronin, The Riemann Zeta Function, de Gruyter, New York, 1992.